Math 8306, Algebraic Topology
Homework 3
Due in-class on **Monday, September 22**

Numbered exercises are from Hatcher's "Algebraic Topology."

1. In class, we defined subdivision maps $s^i_n : \Delta[n+1] \to \Delta[n] \times [0,1]$ for $0 \le i \le n$ by

$$s^i_n(t_1, \cdots, t_{n+1}) = ((t_1, \ldots, \widehat{t_{i+1}}, \ldots, t_{n+1}), t_{i+1}).$$

. Show that these satisfy the relations

   - $s^i_n d^j_{n+1} = \begin{cases} (d^{j-1}_n, id) \circ s^i_{n-1} & \text{if } i < j-1 \\ (d^j_n, id) \circ s^{i-1}_{n-1} & \text{if } i > j \end{cases}$
   - $s^0_n d^0_{n+1} = i_0$
   - $s^n_n d^0_{n+1} = i_1$
   - $s^{i-1}_n d^i_{n+1} = s^i_n d^i_{n+1}$ for $i \ge 1$.

   Use this to show that the operator $h : C_n(X) \to C_{n+1}(X \times [0,1])$ given by

$$h(\sum a_\sigma \sigma) = \sum a_\sigma \sum_{i=0}^{n} (-1)^i (\sigma, id) \circ s^i_n$$

   satisfies $\partial h(x) + h\partial\sigma(x) = i_0(x) - i_1(x)$.

2. Let C_* be the chain complex with

$$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Let D_* be the chain complex with

$$D_n = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

   such that the boundary map $\partial : D_1 \to D_0$ sends m to $2m$.

   Show that the natural projection $\pi : D_* \to C_*$ is a map of chain complexes and it induces the zero map $H_*(D_*) \to H_*(C_*)$. Show that there is no chain homotopy h with $\partial h + h\partial = \pi$ (from π to zero).

3. For $Z \subset Y \subset X$ spaces, show that there is a short exact sequence of singular chain complexes

$$0 \to C_*(Y, Z) \to C_*(X, Z) \to C_*(X, Y) \to 0.$$

What does the resulting long exact sequence of homology groups look like?

4. Hatcher, Exercise 12 on page 132.