Inner products of induced representations

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Here is some motivation.

In a previous lecture, we found the character table for S_4 by starting with the following characters (corresponding to partitions) and reducing it.

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
σ_4	1	1	1	1	1
$\sigma_{3,1}$	4	2	0	1	0
$\sigma_{2,2}$	6	2	2	0	0
$\sigma_{2,1,1}$	12	2	0	0	0
$\sigma_{1,1,1,1}$	24	0	0	0	0

We would like to know *why* this works, and in fact why this method works for any symmetric group. We need to show that

- each row contains a new irreducible representation, not contained in any rows above it, and
- this irreducible representation occurs only *once*.

First, let's recall what the definition of σ_λ was for λ a partition. If λ is a partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, then σ_λ was the character of the permutation representation of S_n acting on the set of all ways to divide n into sets of size $\lambda_1, \lambda_2, \ldots, \lambda_r$.

This representation can be described another way. Let H_λ be the subgroup of S_n consisting of permutations σ such σ preserves the sets $\{1, 2, \ldots, \lambda_1\}$, $\{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\}$, ...

For example, if $n = 4$ and $\lambda = (2, 2)$, then $H_{(2,2)}$ consists of the elements of S_4 that switch the first 2 numbers amongst themselves and switch the last 2 numbers amongst themselves. There are 4 such elements.

$$e, (12), (34), (12)(34)$$

An element of H_λ is determined by

- a permutation of the first λ_1 elements, which is an element of S_{λ_1},
- a permutation of the next λ_2 elements, which is an element of S_{λ_2},

- a permutation of the next λ_3 elements, which is an element of S_{λ_3},
- et cetera.

Therefore, as a group $H_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}$.

Then $\sigma_\lambda = \mathrm{Ind}_{H_\lambda}^{S_n} 1$, the representation induced from the trivial character on H_λ.

Let's define another character τ_λ. The symmetric group S_n has a *sign* representation ρ, which is irreducible of dimension 1. Define

$$\tau_\lambda = \sigma_\lambda \otimes \rho.$$

Since tensor product of representations turns into multiplication of characters, we can write down formulas for the character τ_λ easily. Here they are for S_4.

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
τ_4	1	-1	1	1	-1
$\tau_{3,1}$	4	-2	0	1	0
$\tau_{2,2}$	6	-2	2	0	0
$\tau_{2,1,1}$	12	-2	0	0	0
$\tau_{1,1,1,1}$	24	0	0	0	0

Another way to describe it (using the projection formula from 7.2, Remark 3) is as follows. Let ϵ be the sign representation. Then $\tau_\lambda = \mathrm{Ind}_{H_\lambda}^{S_n} \epsilon$, the representation induced from the sign representation on H_λ.

Now we can finally get to the whole point of this exercise. Let's write down a table that shows the inner products of the representations σ_λ with the τ_λ. (Do some of these computations!)

$\langle -, - \rangle$	$\tau_{1,1,1,1}$	$\tau_{2,1,1}$	$\tau_{2,2}$	$\tau_{3,1}$	τ_4
σ_4	1	0	0	0	0
$\sigma_{3,1}$	4	1	0	0	0
$\sigma_{2,2}$	6	2	1	0	0
$\sigma_{2,1,1}$	12	5	2	1	0
$\sigma_{1,1,1,1}$	24	12	6	4	1

What does this table tell us?

- σ_4 and $\tau_{1,1,1,1}$ share exactly one irreducible representation in common.
- $\sigma_{3,1}$ and $\tau_{2,1,1}$ share exactly one irreducible representation, and $\tau_{2,1,1}$ shares *no* irreducible representations with σ_4.
- $\sigma_{2,2}$ and $\tau_{2,2}$ share exactly one irreducible representation, and $\tau_{2,2}$ shares *no* irreducible representations with σ_4 and $\sigma_{3,1}$.
- et cetera.

So we find the following.

- There is an irreducible representation χ_λ shared by σ_λ and τ_{λ^t}, where λ^t is the conjugate partition.
- This is the *only* representation they share because the inner product is 1; in particular, χ_λ occurs only once in each of them.
- Since τ_{λ^t} has inner product zero with all of the previous characters, and χ_λ occurs in τ_{λ^t}, χ_λ does not occur in any of the previous characters.

This gives us exactly what we wanted.

We'll leave off here for now. What remains is that we need to show why the σ and τ characters have the inner products that they do.