18.905 Problem Set 9

Due Wednesday, November 8 in class

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1. Use naturality to show that if X is the disjoint union of subspaces X_α, then there is an isomorphism of rings
$$H^*(X) \to \prod_\alpha H^*(X_\alpha),$$
where the latter ring has the standard componentwise ring structure
$$(r_\alpha) \cdot (s_\alpha) = (r_\alpha s_\alpha).$$

2. Let X be the lens space $L(n, 1)$ from Hatcher, problem 8 on page 131. (We looked at this space on a previous assignment.) Compute (using the Δ-complex structure when necessary) the cohomology and cup product structure on $H^*(X; \mathbb{Z})$ and $H^*(X; \mathbb{Z}/n)$,

3. Suppose that C_* and D_* are chain complexes. Define a new chain complex $\underline{\mathrm{Hom}}(C_*, D_*)$ which, in degree n, is
$$\underline{\mathrm{Hom}}(C_*, D_*)_n = \prod_p \mathrm{Hom}(C_p, D_{p+n}).$$
In other words, an element of this chain complex in degree n consists of a family of maps $f_p : C_p \to D_{p+n}$ (so n is the amount by which each map raises degree).

   Give a definition of a boundary map $\delta : \underline{\mathrm{Hom}}(C_*, D_*)_n \to \underline{\mathrm{Hom}}(C_*, D_*)_{n-1}$ such that the evaluation map
$$\begin{array}{rcl} \underline{\mathrm{Hom}}(C_*, D_*) \otimes C_* & \to & D_* \\ f \otimes x & \mapsto & f(x) \end{array}$$
is a chain map, where the left-hand side has the standard Leibniz formula for its boundary.

   What does it mean for an element of $\underline{\mathrm{Hom}}(C_*, D_*)_0$ to be a cycle? When do two cycles in $\underline{\mathrm{Hom}}(C_*, D_*)_0$ differ by a boundary?

4. Show the following dual version of the Künneth formula: If C_* is a chain complex where each C_p is levelwise free, and D_* is any chain complex, show that there are natural exact sequences
$$0 \to \prod_{t-s=n+1} \mathrm{Ext}(H_s(C_*), H_t(D_*)) \to H_n(\underline{\mathrm{Hom}}(C_*, D_*)) \to$$
$$\prod_{t-s=n} \mathrm{Hom}(H_s(C_*), H_t(D_*)) \to 0$$
which are split. (Hint: Follow the same method as the proofs of the Künneth formula and the universal coefficient theorems.)