18.906 Problem Set 1

Due Wednesday, February 14 in class

Suppose that $\mathcal{C}$ is a category and Z is an object of $\mathcal{C}$. We then define a contravariant functor $\mathrm{Hom}(-, Z)$ from $\mathcal{C}$ to the category Set of sets as follows: $\mathrm{Hom}(-, Z)$ sends an object Y to the set $\mathrm{Hom}(Y, Z)$, and a map $f : Y \to Y'$ to the map of sets $f^* : \mathrm{Hom}(Y', Z) \to \mathrm{Hom}(Y, Z)$ given by precomposition with f. Such a functor is called a *representable functor*.

1. Show that under this definition, $\mathrm{Hom}(-, Z)$ really is a functor. Given a map $g : Z \to Z'$, show that there is a natural transformation

   $$g_* : \mathrm{Hom}(-, Z) \to \mathrm{Hom}(-, Z')$$

   of functors, and show that $id_* = id$ and $(gh)_* = g_* h_*$. (In other words, $\mathrm{Hom}(-, -)$ is a functor from $\mathcal{C}^{op} \times \mathcal{D}$ to Set.)

2. Conversely, prove that any natural transformation $\theta : \mathrm{Hom}(-, Z) \to \mathrm{Hom}(-, Z')$ is of the form g_* for some unique map $g : Z \to Z'$. (This should look familiar to those who did the assignment on acyclic models last semester.)[1]

3. Recall the definition of a *limit*: Suppose that I is an index category and $F : I \to \mathcal{C}$ is a functor. A limit of I consists of an object $c \in \mathcal{C}$, together with maps $\alpha_i : c \to F(i)$ for all $i \in I$ such that $F(g) \circ \alpha_i = \alpha_j$ for all maps $g : i \to j$ in I. This must satisfy the requirement that any other object d with maps $\beta_i : d \to F(i)$ such that $F(g) \circ \beta_i = \beta_j$ for all g, there exists a unique map $\gamma : d \to c$ such that $\alpha_i \circ \gamma = \beta_i$ for all $i \in I$. We usually write $c = \lim_I F$.

   Show that this is *equivalent* to providing a natural isomorphism of functors

   $$\mathrm{Hom}(-, c) \to \lim_I \mathrm{Hom}(-, F(i)).$$

4. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories, and F is a functor from $\mathcal{C}$ to $\mathcal{D}$. We say that a functor G from $\mathcal{D}$ to $\mathcal{C}$ is *right adjoint* to F if there is a natural isomorphism of functors

   $$\eta : \mathrm{Hom}_{\mathcal{D}}(F(-), -) \to \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

   from $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$.[2] Use the previous questions to show that any two right adjoints of F are naturally isomorphic.

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[1] Questions 1 and 2 constitute what is usually known as the Yoneda lemma.
[2] Strictly speaking, the natural isomorphism η is part of the data of a right adjoint.