18.906 Problem Set 11

Due Wednesday, May 2 in class

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1. When does the tangent bundle of $\mathbb{RP}^n$ have trivial Stiefel-Whitney classes?

2. Suppose ξ, ξ' are vector bundles on X. Show that

$$\bigwedge^n (\xi \oplus \xi') \cong \bigoplus_{p+q=n} (\wedge^p \xi) \otimes (\wedge^q \xi')$$

   for $n > 1$. Use this formula with ξ' the trivial bundle to extend these to natural operations $\lambda^n : KO(X) \to KO(X)$ for X a finite CW-complex.

3. For $E \in KO(X)$, define an associated characteristic power series $\lambda_t(E) = \sum_i \lambda^i(E)t^i \in KO(X)[[t]]$. Deduce that $\lambda_t(E + E') = \lambda_t(E)\lambda_t(E')$.

   Deduce that when ξ is a direct sum of line bundles $L_1, \ldots, L_n$, the elements $\lambda^i([\xi])$ are the elementary symmetric polynomials in the elements $[L_i]$ of $KO(X)$.

4. If $p(x_1, x_2, \ldots, x_n)$ is a polynomial with integer coefficients, we can define an operation on θ_p on $KO(X)$ by

$$E \mapsto p(\lambda^1(E), \lambda^2(E), \ldots, \lambda^n(E)).$$

   Show that, for each n, there is such a polynomial p in $x_1, \ldots, x_n$ with operation $\psi^n = \Theta_p$ such that whenever we have $E = E_1 + \cdots + E_m$ where each E_i is the class associated to a line bundle, then

$$\psi^n(E) = E_1^n + E_2^n + \cdots + E_m^n.$$

   (Hint: Construct p using symmetric polynomials.)

(Note: Questions 2 through 4 have, of course, analogous versions for $K(X)$.)