18.906 Problem Set 12 (Final)

Due Wednesday, May 9 in class

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1. Suppose that you are given a sequence of based spaces E_n for $n \geq 0$ together with based maps $\sigma_n : E_n \to \Omega E_{n+1}$ which are isomorphisms on homotopy groups. Show that σ_n gives rise to a natural isomorphism $[X, E_n]_* \to [\Sigma X, E_{n+1}]_*$ for all n. Explain how this allows us to define

$$\underline{E}^n(X) = [\Sigma^k X, E_{n+k}]_*$$

for all $n \in \mathbb{Z}$, X a CW-complex. Show that $\underline{E}^*(-)$ is a generalized (reduced) cohomology theory on CW-complexes in the sense that it is a collection of functors which are homotopy invariant and abelian group valued, satisfy CW-excision, and have a long exact sequence associated to a cofibration $A \subset X$.[1]

2. Use the Adem relations to show that the mod-2 Steenrod algebra is generated by the elements $Sq^{2^k}, k \geq 0$. Conclude that there can only be a space X with cohomology ring $H^*(X; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^3)$ if x is in degree 2^k for some k.

3. Determine the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2.

4. Use the splitting principle, together with the identities satisfied by Steenrod squares, to find formulas for $Sq^i(w_k(\xi))$ for all $i \geq 0$, $k \leq 3$, where $w_k(\xi)$ is the k'th Stiefel-Whitney class of a vector bundle ξ on a space X.[2]

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[1]Such a collection of spaces and structure maps is called an Ω-*spectrum*; if we drop the condition that the structure maps are isomorphism on homotopy groups, we get a *spectrum*. (Depending on who you ask, this terminology is a little outdated.)

[2]The results of this exercise imply that the third Stiefel-Whitney class w_3 is determined by w_1, w_2, and the action of the Steenrod algebra on $H^*(X; \mathbb{Z}/2)$. In fact, the only Steifel-Whitney classes which are "primitive", in the sense of not being determined by the lower classes and the Steenrod algebra, are the classes w_{2^k}.