18.906 Problem Set 7

Due **Friday**, April 6 in class

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1. Suppose that $G \to H$ is a continuous homomorphism of topological groups and $P \to X$ is a principal G-bundle. Show that there is a principal H-bundle $P' = P \times_G H \to X$.

   Let F be a space with a left action of H, given a left action of G by restriction. Show that the show that the fiber bundles $P \times_G F \to X$ and $P' \times_H F \to X$ are isomorphic.

2. Suppose G acts on a space F on the left such that the map $g_* : H_*(F) \to H_*(F)$ is the identity for all $g \in G$. Let $P \to X$ be a principal G-bundle, and let $E \to X$ be the associated fiber bundle $P \times_G F \to X$ with fiber F. Show that the action of $\pi_1(X)$ on $H_*(F)$ is trivial.

   In particular, if the group G is path-connected, show that the action of G on $H_*(F)$ is *always* trivial.

3. Suppose that $\xi \to X$ is a complex vector bundle with inner product associated to a principal $U(n)$-bundle $P \to X$. There is then a unit sphere bundle $S \subset \xi$ consisting of the unit vectors; this is a fiber bundle over X with fiber S^{2n-1}.

   Show that $U(n)$ acts trivially on $H_*(S^{2n-1})$. Use the Serre spectral sequence in cohomology to define an interesting element in $H^{2n}(X)$ and explain how it determines the differential $d_{2n} : H^p(X; H^{2n-1}(S^{2n-1})) \to H^{p+2n}(X)$.

4. Suppose that L is a generalized homology theory, i.e.:

   - We have a collection of functors L_n for $n \in \mathbb{Z}$ from the category of pairs $\{(X, A) | A \subset X\}$ to the category of abelian groups.
   - We have natural boundary maps
     $$\partial : L_n(X, A) \to L_{n-1}(A) = L_{n-1}(A, \emptyset).$$
   - The sequence of maps
     $$\cdots \to L_{n+1}(X, A) \to L_n(A) \to L_n(X) \to L_n(X, A) \to L_{n-1}(A) \to \cdots$$
     is exact for any $A \subset X$.
   - If f and g are two maps of pairs $(X, A) \to (Y, B)$ which are homotopic through maps of pairs, then the maps f_* and g_* from $L_n(X, A) \to L_n(Y, B)$ are the same.
   - If $V \subset A$ is a subspace such that the closure of V is contained in the interior of A, then the map $L_n(X \setminus V, A \setminus V) \to L_n(X, A)$ is an isomorphism.

- If X is a disjoint union of disconnected subspaces X_α, then $L_n(X) = \oplus_\alpha L_n(X_\alpha)$.

For short, we write $L_n = L_n(*)$ for the coefficients. One can show using the long exact sequence that that $L_n(S^k, pt) \cong L_{n-k}$.

Show that a degree d map of based spaces $S^k \to S^k$ induces multiplication by d on $L_n(S^k, pt)$. (Hint: Any such map is homotopic to a pinch map $S^k \to \vee_d S^k$ followed by the fold map $\vee_d S^k \to S^k$.)

If X is a finite CW-complex, use the filtration of X by skeleta to get a collection of long exact sequences and assemble these into an exact couple as follows.

$$\begin{array}{ccc} \bigoplus_{p,q} L_{p+q}(X^{(p-1)}) & \longrightarrow & \bigoplus_{p,q} L_{p+q}(X^{(p)}) \\ & \nwarrow \quad \swarrow & \\ & \bigoplus_{p,q} L_{p+q}(X^{(p)}, X^{(p-1)}) & \end{array}$$

Express the E_1-term and the d_1-differential in terms of the cellular chain complex of X. Use this to compute the E_2-term.

(This spectral sequence converges to $L_*(X)$; the spectral sequence is called the *Atiyah-Hirzebruch* spectral sequence.)