Math 8245 **Homework 1, due Monday 29 September** **PJW**

GAP questions:

1. Let $p_1, p_2, p_3, \ldots = 2, 3, 5, \ldots$ be the sequence of primes, and define
$$E_n = p_1 \cdot p_2 \cdots p_n + 1.$$
These are the numbers which appear in Euclid's proof that there are infinitely many primes, and have the property that E_n is not divisible by any of $p_1, \ldots, p_n$. Write a program in GAP which prints out which of the numbers E_n are prime, where $1 \leq n \leq 80$.
[This exercise tests use of loops and the GAP functions for integers. Instead of typing in your program live within a GAP session, you could try creating the program in a separate file in the directory from which you started GAP, and read in the file to GAP using Read("filename"); That way you do not need to type everything again when you make corrections.]

2. Consider the groups:
$$\begin{aligned}
g1 &= \langle (1,2,3,4,5,6,7,8), (2,4)(3,7)(6,8) \rangle \\
g2 &= \langle (1,2,3,4,5,6,7,8), (2,8)(3,7)(4,6) \rangle \\
g3 &= \langle (1,2,3,4,5,6,7,8), (2,6)(4,8) \rangle \\
g4 &= \langle (1,4,6,8,10,12,14,15)(2,3,5,7,9,11,13,16), \\
&\qquad (1,2)(3,15)(4,16)(5,14)(6,13)(7,12)(8,11)(9,10) \rangle \\
g5 &= \langle (1,2,3,4,5,6,7,8), (9,10) \rangle.
\end{aligned}$$
(a) For each group compute
(i) a list of the elements of the group,
(ii) a list of the orders of the elements of the group.
(b) Determine whether any of these groups are isomorphic to one another.
[Computer algorithms which establish an isomorphism between two groups are very poor — they basically run through all possible bijections and see if any of them are group homomorphisms. Instead, to show that certain groups are isomorphic here you should identify the groups in question as groups you already know something about, and use some theory to establish isomorphism.]
(c) In the case of the group $g1$, compute the lattice of subgroups. Show that $g1$ has a subgroup which is cyclic of order 8, another subgroup which is dihedral of order 8, and a third subgroup which is quaternion of order 8. In each case provide generators for the subgroup in question. Draw a picture of the lattice of subgroups, where one subgroup is shown immediately below another if one is a maximal subgroup of the other — in other words, draw the Hasse diagram.

[There are commands `ConjugacyClassesSubgroups` and `LatticeSubgroups` which I am sure some of you would be tempted to explore in tackling this problem. I suggest that it would be at least as easy for you to construct the lattice of subgroups by intelligent direct use of the functions I have already shown you in GAP. If you do insist on finding out about the subgroup functions I have just mentioned, be warned that I ask for a lattice of subgroups, not of conjugacy classes of subgroups, so I want all subgroups in the picture. Also, I ask for a picture, not the kind of output which these built-in functions produce.]

Extra questions: Do **not** hand in.

3. Write a function `AllOnes(n)` which given the non-negative integer n returns the integer 11...111 with n 1s. The syntax for functions is

```
allones:=function(n)
    local ;
    ...
    return(output);
end;
```

4. Suppose you are given a function `Conway` which when applied to a list of integers such as `[1,3,3,2]` returns `[1,1,2,3,1,2]`, for example. Write a routine in GAP which prints out the first 10 iterates of Conway applied to a list, e.g. applied to `[1,3,3,2]` it will return the 10 lists

```
[1,3,3,2]
Conway([1,3,3,2])
Conway(Conway([1,3,3,2]))
...
```

Bear in mind that within `Print` the string `"\n"` forces a line break before the next output is printed.

Math 8245 **Homework 2, due Monday 13 October** **PJW**

GAP questions:

1. Use GAP to show that

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 = 1\rangle \cong A_5 \times C_2.$$

2. Use GAP to show that $SL(2,5)$ has a normal subgroup of order 2 such that the quotient is isomorphic to A_5. Show that $SL(2,5)$ has no subgroup isomorphic to A_5. Identify the Sylow 2-subgroups of $SL(2,5)$.

3. The generalized quaternion group of order 2^n has a presentation

$$\langle a, b \mid a^{2^{n-1}} = 1,\, b^2 = a^{2^{n-2}},\, bab^{-1} = a^{-1}\rangle.$$

Use GAP to investigate the generalized quaternion group of order 32. Get a list of the orders of the elements. Compute the derived subgroup and the center. Draw a picture of the lattice of subgroups of this group. What is the minimum degree of a faithful permutation representation of this group?

4. Investigate similarly the groups

$$g1 = \langle a, b, c \mid a^3 = b^3 = c^3 = [a, b] = 1,\, cac^{-1} = ab,\, cbc^{-1} = b\rangle,$$
$$g2 = \langle a, b \mid a^9 = b^3 = 1,\, bab^{-1} = a^4\rangle.$$

I think the lattices of subgroups are too big to be worth doing, and they do not give much insight. Do you agree?

Extra questions: Do **not** hand in.

5. Let s be the Sylow 2-subgroup of the group

$$g = \langle(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11), (3, 7, 11, 8)(4, 10, 5, 6)\rangle.$$

a) Obtain a permutation representation of s on 8 symbols.
b) It is the case that s is isomorphic to one of the groups in question 2. To which one is it isomorphic?
c) Construct a subgroup of g of order divisible by 2, which is not a 2-group and which does not contain a Sylow 2-subgroup of g (any such subgroup will do!).
[This group g is the Mathieu group M_{11}. Use SylowSubgroup, Orbits, Operation.]

6. Repeat question 3. with the group

$$\langle (1,2,3,4,5,6,7,8,9,10,11,12,13), (2,3)(5,10)(7,11)(9,12) \rangle.$$

[This group is $PSL(3,3)$.]

7. Let g be the group generated by the four permutations

$$\begin{aligned}
&(1,2),\\
&(1,3)(2,4),\\
&(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16), \text{ and}\\
&(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16).
\end{aligned}$$

a) Show that this group is not isomorphic to the Sylow 2-subgroup of A_{16}.
b) How many properties of these two groups can you find which would be the same if the groups were isomorphic, and in this instance are different?
[This problem arose some time ago in discussions between Professors Feshbach and Lannes.]

Math 8245 **Homework 3, due Monday 27 October** **PJW**

GAP questions:

1. The Mathieu group M_{12} may be generated by permutations

$$(1,2,3,4,5,6)(7,8,9,10,11,12) \quad \text{and} \quad (1,9,12,7,11)(6,2,8,3,5).$$

a) Make a stabilizer chain for M_{12} and determine what the orbits $\Delta^{(i)}$ are.
b) What is the smallest possible size of a base that a group of size $|M_{12}|$ acting on 12 points can have?
c) For each i in $\{1,\ldots,12\}$ find a word in the generators of M_{12} which sends 1 to i. [You may wish to modify the function `righttransversal` which formed part of the code available in Lesson 6 so that it produces a word in the generators, or it is probably faster to do it by hand.]
d) The stabilizer of 1 in M_{12} is M_{11}. Compute a set of generators for $\mathrm{Stab}_{M_{12}}(1)$, expressing them as words in the generators of M_{12}.

2. Suppose that `grp` is a permutation group and that `sc:=StabChain(grp)` is a stabilizer chain for the group. The following is an attempt to write a function `iselement` of arguments `sc` and `g` which returns `true` precisely when `g` is a member of the permutation group `grp`.

```
iselement:=function(sc,g)
  local h,i;
  if sc.generators=[] then return g=();
  fi;
  h:=g;
  while sc.orbit[1]^h<> sc.orbit[1] do
    h:=h*sc.transversal[1^h];
  od;
  iselement(sc.stabilizer,h);
end;
```

There are some deliberate mistakes and omissions in the code. Make corrections to the code so that it works, correctly determining whether each of the permutations $(1,2,3,7,11)(4,9,6,10,5)$ and $(1,12)$ belong to

$$M_{11} = \langle (1,2,3,4,5,6,7,8,9,10,11),(3,7,11,8)(4,10,5,6)\rangle.$$

Answer the following questions:
(a) Why is the line `h:=g` present in the code? Is it necessary?
(b) Explain why it is that after some small changes of a typagrophical natrue the code will run without errors, but does not produce any answer. What should be done to correct this?

Extra questions:

3. Write a function `OrbitInfo:=function(grp,i)` whose arguments are a permutation group `grp` and an integer `i`, which returns a list `[a,b]` where `a` is a list starting with `i` whose entries are the orbit containing `i` and where `b` is a list whose entries are either undefined or are taken from the given generators of `grp`, with the property that `b[j]` is defined if and only if `j` is in the same orbit as `i`, and then `i^b[j]` either appears earlier in `a` or is `a[1]`.
(You are thus asked to produce a list `[sc.orbit,sc.transversal]` where `sc` is a stabilizer chain. However, I would like you to write code for yourself, starting from scratch.)

4. The Mathieu group M_{24} may be generated by permutations

$$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14,15,16,17,18,19,20,21)(22,23)$$

and

$$(1,2,5,7,15,20,14,23,21,11,16,19,24,6,8,4,17,3,10,13,18)(9,22,12).$$

a) Make a stabilizer chain for M_{24} and determine the lengths of the orbits $\Delta^{(i)}$.
b) What is the smallest size of a base for a group of size $|M_{24}|$ acting on 24 points?
[I found the following function useful:

```
orbitlengths:=function(sc)
  local a;
  a:=ShallowCopy(sc);
  while IsBound(a.stabilizer) do
    Print("Orbit of length ", Length(a.orbit), "\n");
    a:=ShallowCopy(a.stabilizer);
  od;
  return;
end;
```

I used `ShallowCopy` a couple of times. Was it strictly necessary?]

GAP questions:

1. (Question 2 from page 114 of Johnson's book.) Prove in detail that in enumerating cosets for the presentation

$$T_n = \langle x, y \mid x^n y^{n+1}, x^{n+1} y^{n+2} \rangle, \quad n \in \mathbb{N},$$

   at least $n+1$ symbols are needed. Can you enlarge this lower bound?

2. Let `g` denote the group with presentation given in question 1. When $n = 3$ the command
   `gap> CosetTable(g,Subgroup(g,[]));`
   succeeds in showing that there is only one coset, but when $n = 20$ it fails.
   (i) Find the least value of n for which this command fails to enumerate the cosets, without increasing the default number of cosets allowed.
   (ii) Investigate what is going on and give an explanation, given that it is possible to show by coset enumeration that there is only one coset by introducing far fewer cosets than the default maximum. Does GAP simply make poor choices of cosets which it introduces?